
Geometric Matrix Completion: A Functional View

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Abstract

We propose a totally functional view of geometric matrix completion problem. Differently from existing work, we propose a novel regularization inspired from the functional map literature that is more interpretable and theoretically sound. On synthetic tasks with strong underlying geometric structure, our framework outperforms state of the art by a huge margin (two order of magnitude) demonstrating the potential of our approach. On real datasets, we achieve state-of-the-art results at a fraction of the computational effort of previous methods. Our code is publicly available at https://github.com/Not-IITian/functional_matrix_completion

1. Introduction

Matrix completion deals with the recovery of missing values of a matrix of which we have only measured a subset of the entries,

$$\text{Find } \mathbf{X} \text{ s.t. } \mathbf{X} \odot \mathbf{S} = \mathbf{M} \odot \mathbf{S}. \quad (1)$$

Here \mathbf{X} stands for the unknown matrix, $\mathbf{M} \in \mathbb{R}^{m \times n}$ for the ground truth matrix, \mathbf{S} is a binary mask representing the input support, and \odot denotes the Hadamard product. In general, without any constraints, this problem (1) is ill-posed and not solvable. However if the rank of underlying matrix is small, the number of degrees of freedom decreases and thus, it is common to find the lowest rank matrix that agrees with known measurements. Under this low rank assumption, the matrix completion problem can be rewritten as,

$$\min_{\mathbf{X}} \text{rank}(\mathbf{X}) + \frac{\mu}{2} \|(\mathbf{X} - \mathbf{M}) \odot \mathbf{S}\|_F^2. \quad (2)$$

Various problems in collaborative filtering can be posed as a *matrix completion* problem (Kalofolias et al., 2014; Rao et al., 2015), where for example the columns and rows represent users and items, respectively, and matrix values represent a score determining whether a user would like an item or not. This setting was particularly popularized by

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the Netflix challenge (Koren et al., 2009). Often, additional structural information is available in the form of column and row graphs representing similarity of users and items, respectively. Such geometric information is not exploited by rank based prior work that only seeks a purely algebraic solution by optimizing for low rank (Candès and Recht, 2009). Prior work that incorporates geometric structure into matrix completion problems (Monti et al., 2017) obtains state-of-the-art results using powerful pattern extraction ability of graph CNN but falls short of giving a principled framework to model such geometric information.

(Boyarski et al., 2020) makes an attempt to build a principled framework that is based on a functional map representation (Ovsjanikov et al., 2012) and also competes empirically with highly engineered models such as multi-graph CNN (Monti et al., 2017). One advantage of working with the functional map representation is that its size, and thus the number of unknowns, is typically much smaller, and only controlled by the size of the basis, independent of the number of nodes in graphs. Although (Boyarski et al., 2020) obtains state-of-the-art results on both synthetic and real datasets, it introduces several non-convex regularization terms thereby making the overall problem harder to optimize. To address this challenge, we propose a simple formulation based on functional map, consisting of a single regularizer, that mitigates the problems associated with (Boyarski et al., 2020).

Contributions. Our contributions are threefold. First, we propose a novel functional view of the geometric matrix completion problem that is convex and theoretically grounded. Second, our method, on synthetic tasks with strong underlying geometric model, sets new benchmarks in modelling the geometric information with over 100 times improvement over prior work. Third, our proposed model obtains state-of-the-art results on various real world recommendation datasets while being more intuitive and simpler to optimize and thereby easier to analyze and reproduce.

2. Related work

Geometric matrix completion. There exist several prior works on geometric matrix completion problem that exploit geometric information (Berg et al., 2017; Kalofolias et al., 2014; Rao et al., 2015) such as graphs encoding relations between rows and columns. More recent work leverages

deep learning on geometric domains (Berg et al., 2017; Monti et al., 2017) to extract relevant information from geometric data such as graphs. As argued in (Boyarski et al., 2020), while these techniques achieve state-of-the-art results, their design is highly engineered and thus, non-intuitive.

Functional Maps. Our work is mainly inspired from the functional map framework (Ovsjanikov et al., 2012) used ubiquitously in non-rigid shape correspondence, and also extended to handle challenging partial matching cases, e.g. (Litany et al., 2017). This framework has recently been adapted for geometric matrix completion in (Boyarski et al., 2020), where the authors propose to build a functional map between graphs of rows and columns. As noted in several works, isometry between two spaces is a key to functional map representation. Assuming isometry between real world graphs is, however, over optimistic. Thus, one way to work under relaxed isometry condition is to instead align the eigen-basis with additional transformation matrix while promoting its diagonal structure (Litany et al., 2017). (Boyarski et al., 2020) achieve this with a range of transformations on the eigen-basis of the graph Laplacians. However, they 1) impose several regularization terms each with a scaling hyperparameter and some even with different initializations 2) explore a huge range of hyperparameter space and question remains on why does it work so well.

3. Preliminaries

In this section, we cover some preliminaries about product graphs and functional maps.

Product graphs Let $\mathcal{G} = (V, E, W)$ be a (weighted) graph with its vertex set V and edge set E and adjacency matrix denoted by W . Graph Laplacian L is given by $L = D - W$, where $D = \text{diag}(W)$ is the *degree matrix*. L is symmetric and positive semi-definite whenever the weights are non-negative, and as such admits a spectral decomposition $L = \Phi \Lambda \Phi^\top$. It is well-known that the spectrum of the Laplacian contains structural information about the graph (Spielman, 2009). Let $\mathcal{G}_1 = (V_1, E_1, W_1)$, $\mathcal{G}_2 = (V_2, E_2, W_2)$ be two graphs, with $L_1 = \Phi \Lambda_1 \Phi^\top$, $L_2 = \Psi \Lambda_2 \Psi^\top$ being their corresponding graph Laplacians. The basis Φ, Ψ can be used to represent functions on these graphs. We define the Cartesian product of \mathcal{G}_1 and \mathcal{G}_2 , denoted by $\mathcal{G}_1 \square \mathcal{G}_2$, as the graph with vertex set $V_1 \times V_2$, on which two nodes $(u, u'), (v, v')$ are adjacent if either $u = v$ and $(u', v') \in E_2$ or $u' = v'$ and $(u, v) \in E_1$.

Functional maps. Let X be a function defined on $\mathcal{G}_1 \square \mathcal{G}_2$. It can be encoded as a matrix of size $|V_1| \times |V_2|$ and represented using the bases Φ, Ψ of the individual graph Laplacians, $C = \Phi^\top X \Psi$. In the shape processing com-

munity, such C is called a *functional map*, as it is a map between the functional spaces of \mathcal{G}_1 and \mathcal{G}_2 . One of the advantages of working with the functional map representation C rather than the matrix X is that its size is typically much smaller, and only controlled by the size of the basis, independent of the number of nodes in \mathcal{G}_1 and \mathcal{G}_2 , resulting in simpler optimization problems. Moreover, the projection onto a basis also provides a strong regularization, which can itself be beneficial for both shape matching, and, as we show below, for matrix completion as well. For example, given two functions, $x = \Phi \alpha$ on \mathcal{G}_1 and $y = \Psi \beta$ on \mathcal{G}_2 , one can use C to map between their representations α and β , i.e., $\alpha = \Phi^\top x = C \Psi^\top y = C \beta$.

4. Functional Geometric Matrix Completion

We assume that we are given a set of samples from some unknown matrix $M \in \mathbb{R}^{m \times n}$, along with a binary indicator mask S that is 1 for measured samples and 0 for missing ones. In addition, we are given two graphs $\mathcal{G}_r, \mathcal{G}_c$, encoding relations between the rows and the columns, respectively. We represent the Laplacians of these graphs and their spectral decompositions by $L_r = \Phi \Lambda_r \Phi^\top$, $L_c = \Psi \Lambda_c \Psi^\top$. We minimize the objective function of the following form:

$$\min_{\mathbf{X}} E_{\text{data}}(\mathbf{X}) + \mu E_{\text{reg}}(\mathbf{X}) \quad (3)$$

with E_{data} denoting a data term of the form

$$E_{\text{data}}(\mathbf{X}) = \|(\mathbf{X} - M) \odot S\|_F^2, \quad (4)$$

As observed in (Boyarski et al., 2020), we can decompose $\mathbf{X} = \Phi C \Psi^\top$. Remarkably, the data term itself, as we show in our experiments later, when expressed through the functional map i.e. $\mathbf{X} = \Phi C \Psi^\top$ already recovers low-rank matrices and outperforms the approach of (Boyarski et al., 2020) on synthetic geometric experiments. Before we explain the choice and motivation of our regularizer E_{reg} , we explain next why the data term itself already works remarkably well on rank-constrained geometric problems.

4.1. Low Rank Geometric Matrix Completion

Our first observation is that by using a reduced basis to represent a function \mathbf{X} on the product space $\mathcal{G}_1 \square \mathcal{G}_2$ already provides a strong regularization, which can be sufficient to recover a low rank matrix from a sparse signal.

Specifically, suppose that we constrain \mathbf{X} to be a matrix such that $\mathbf{X} = \Phi C \Psi^\top$ for some matrix C . Note that if Φ and Ψ have k columns each then C must be a $k \times k$ matrix. We would like to argue that solving Eq. (4) under the constraint that $\mathbf{X} = \Phi C \Psi^\top$ will recover the underlying ground truth signal \mathbf{Y} if it is low rank and satisfies an additional condition that we call basis consistency.

For this suppose that the ground truth hidden signal M has rank r . Consider its singular value decomposition $M = U\Sigma V^\top$. If M has rank r , then Σ is a diagonal matrix with r non-zero entries. We call M *basis-consistent* with respect to Φ, Ψ if the first r left singular vectors U_r (i.e., those corresponding to non-zero singular values) lie in the span of Φ , and the first r right singular vectors V_r lie in the span of Ψ . In other words, there exist some matrices R, Q s.t. $U_r = \Phi R$ (note that this implies $k \geq r$) and $V_r = \Psi Q$.

Given this definition, it is easy to see that all basis-consistent matrices with rank $r \leq k$ can be represented by some functional map C . In other words, given Y that is basis-consistent, there is some functional map C s.t. $Y = \Phi C \Psi^\top$. Conversely any $X = \Phi C \Psi^\top$ has rank at most k and must be basis-consistent by construction.

Second, suppose we are optimizing Eq (4) under the constraint $X = \Phi C \Psi^\top$ and that the optimum, i.e., the ground truth matrix M , is basis-consistent. Then since the energy $E(C)$ is convex and if there are enough known samples to fully constrain the corresponding linear system, then we are guaranteed to recover the optimum low-rank basis-consistent matrix. We note briefly that the argument above can also be made approximate, when the ground truth matrix is not exactly, but only approximately basis consistent, by putting appropriate error bounds.

This simple observation suggests that by restricting $X = \Phi C \Psi^\top$ and optimizing over the matrix C instead of X , already provides a strong regularization that can help recover appropriate low-rank signals even without any other regularization. In practice, we observe that a weak additional regularization is sufficient to obtain state-of-the-art results.

4.2. Functional Regularization

For clarity, we first describe the regularization terms introduced in (Boyarski et al., 2020) briefly

$$E_{\text{reg}} = \mu_r E_{\text{dir}}^r + \mu_c E_{\text{dir}}^c + \rho_c E_{\text{diag}}^c + \rho_r E_{\text{diag}}^r \quad (5)$$

E_{dir}^r is the Dirichlet energy of X on row graph, given by $E_{\text{dir}}^r(X) = \text{tr}(X^\top L_r X)$. A similar term is used for column graph $E_{\text{dir}}^c(X) = \text{tr}(X L_c X^\top)$. In addition to the Dirichlet energy, (Boyarski et al., 2020) also introduces two regularization on the transformation matrix P, Q . As described earlier, the purpose of these transformations is to rotate the original eigen basis $\Phi \Psi$ that will simplify the structure of C . $E_{\text{diag}}^r(P) = \|\text{off}(P^\top \Lambda_r P)\|_F^2$, where $\text{off}(\cdot)$ denotes the off-diagonal elements. A similar treatment to the columns graph gives $E_{\text{diag}}^c(Q) = \|\text{off}(Q^\top \Lambda_c Q)\|_F^2$. We briefly mention that in addition to SGMC, (Boyarski et al., 2020) also proposes a multi-resolution spectral loss named SGMC-Zoomout (SGMC-Z) (Melzi et al., 2019) with its own hyperparameters besides the four hyperparameters

mentioned in Eq. (5).

In contrast to SGMC or SGMC-Z, our E_{reg} contains a single regularization term on the functional map induced between row space and column space as described next.

Laplacian Commutativity as a Regularizer Our main idea is to use the simplest possible regularizer, which furthermore leads to a convex optimization problem and can achieve state-of-the-art results. For this, we borrow a condition that is prominent in the functional map literature (Ovsjanikov et al., 2016). Namely, in the context of surfaces, the functional map is often expected to *commute with the Laplace-Beltrami operator*:

$$E_{\text{reg}} = \|\mathbf{C} \Lambda_r - \Lambda_c \mathbf{C}\|^2, \quad (6)$$

where Λ_r and Λ_c are diagonal matrices of Laplacian eigenvalues of the source and target.

For shape matching problems, this constraint helps to find better mappings because functional maps that commute with the Laplacian must arise from near isometric point-to-point correspondences (Rosenberg, 1997; Ovsjanikov et al., 2012). More broadly, commutativity with the Laplacian imposes a diagonal structure of the functional map, which intuitively promotes preservation of low frequency eigenfunctions used in the basis. In the context of matrix completion, this can be interpreted as approximate preservation of global low frequency signals defined on the two graphs.

Given these above definitions, our problem defined in Eq. (3) reduces to

$$\min_C \|(X - M) \odot S\|_F^2 + \|\mathbf{C} \Lambda_r - \Lambda_c \mathbf{C}\|^2 \quad (7)$$

where $X = \Phi C \Psi^\top$

In practice we observe faster convergence if we replace C with PCQ^\top and let all three matrices P, C, Q be free variables.

4.3. Implementation

The optimization is carried out using gradient descent in Tensorflow (Abadi et al., 2015). Similar to (Boyarski et al., 2020), we initialize both P and Q with an identity matrix with size equal to that of underlying matrix M corresponding to respective dataset and C by projecting $X \odot S$ on the first eigen vector of L_c and L_r . Our formulation contains two hyperparameters: namely the size of C and the weighing scalar μ . We divide the number of available entries in the matrix randomly into training and validation set in a 95 to 5 ratio respectively. We set μ to be .00001 and learning rate to .000001 for all the experiments. The size of C is different for different datasets and set according to the validation performance on each dataset.

Table 1. Testing dependence on the rank of underlying matrix

Rank	Ours	Ours-FM	SGMC
5	1e-7	2e-5	1e-4
10	2e-7	2e-5	2e-4
12	5e-7	4e-5	9e-4
15	6e-3	1e-3	1e-2
20	3e-2	1e-2	5e-2

Table 2. Testing dependence on the density of sampling set.

Density	Ours	Ours-FM	SGMC
1	2e-2	2e-2	1e-1
5	8e-7	1e-3	5e-4
10	2e-7	5e-5	2e-4
20	1e-7	2e-5	1e-4

5. Results

5.1. Experimental study on synthetic dataset

For a fair comparison with (Boyarski et al., 2020), we use the graphs taken from the synthetic Netflix dataset. Synthetic Netflix is a small synthetic dataset constructed by (Kalofolias et al., 2014) and (Monti et al., 2017), in which the user and item graphs have strong community structure. It is useful in conducting controlled experiments to understand the behavior of geometry-exploiting algorithms. In all our tests, we use a randomly generated low rank matrix on the product graph $\mathcal{G}_c \square \mathcal{G}_r$.

Baselines **Ours-FM**: This baseline only optimizes for C without any regularization. All results are obtained with C of size 30×30 . **SGMC**: All results are obtained with their open source code with their optimal parameters.

Test Error. To evaluate the performance of the algorithms in this section, we report the *root mean squared error*, computed on the complement of the training set. We compare the two approaches on different constraints as follows:

Rank of the underlying matrix. We explore the effect of the rank of the underlying matrix, showing that as the rank increases upto 15 or 20, it becomes harder for both methods to recover the matrix. As the rank increases, the reconstruction error increases, but it increases slower for us than for SGMC. We remark that Ours-FM consistently outperforms SGMC for all rank.

Sampling density. We investigate the effect of the number of samples on the reconstruction error. We demonstrate that in the data-poor regime, our regularization is strong enough to recover the matrix, compared to the performance achieved by incorporating geometric regularization through SGMC. These experiments are summarized in Table 2.

Table 3. Test error on Synthetic Netflix, Flixster (Jamali and Ester, 2010), and Movielens-100K (Harper and Konstan, 2016)

Model	Synthetic Netflix	Flixster	ML-100K
MC Candès and Recht (2009)	–	1.533	0.973
GMC Kalofolias et al. (2014)	0.3693	–	0.996
GRALS Rao et al. (2015)	0.0114	1.245	0.945
RGCNN Monti et al. (2017)	0.0053	0.926	0.929
GC-MC Berg et al. (2017)	–	0.917	0.910
Ours-FM	0.0064	1.02	1.12
DMF Arora et al. (2019)	0.0468	1.06	0.922
SGMC	0.0021	0.900	0.912
SGMC-Z	0.0036	0.888	0.913
Ours	0.0022	0.888	0.915

5.2. Results on recommender systems datasets

In addition to synthetic Netflix, we also validate our method on two more recommender systems datasets for which row and column graphs are available in Table 3. We compare with following baselines:

SGMC(Z): In addition to SGMC, (Boyarski et al., 2020) also proposed a multi resolution spectral loss named SGMC-Zoomout (Melzi et al., 2019). **DMF**: This is a matrix factorization approach that was adapted for matrix completion tasks by (Boyarski et al., 2020). Note that this approach does not incorporate any geometric information. **Ours-FM**: This method only optimizes the data term, over C , without any regularization. All results are obtained with C of size 30×30 . We explain several observations from Table 3: First, our baseline, Ours-FM, obtains surprisingly good performance across datasets. This underscores the regularization brought in by the Laplacian eigen basis of row and column graphs. Second, the non geometric model such as DMF shows competitive performance with all the other methods on ML-100K. This suggests that the geometric information is not very useful for this dataset. Third, our proposed algorithm is competitive with the other methods while being simple and interpretable. On Synthetic Netflix, we obtain best results with randomization on the underlying graph structure.

6. Conclusion

We propose a functional view for geometric matrix completion and establish empirically and theoretically that using a reduced basis to represent a function on the product space of two graphs already provides a strong regularization, which is sufficient to recover a low rank matrix approximation from a sparse signal. Moreover, we propose a novel regularization and show, through extensive experiments on real and synthetic datasets, that our single regularization is very competitive when compared to the combination of several different regularizations proposed before.

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